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T.A. CHAPMAN

CANONICAL EXTENSIONS OF HOMEOMORPHISMS

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Contents

Canonical Extensions Of Homeomorphisms

by

T.A. Chapman *)

1. Introduction

Let the Hilbert cube Q be denoted by $\Pi_{i=1} I_i$, where each I_i is the closed interval [-1,1], and let $s = \Pi_{i=1} I_i^0$, where each I_i^0 is the open interval (-1,1). By a Hilbert cube manifold (or Q-manifold) we mean a separable metric space having an open cover by sets homeomorphic to open subsets of Q and by a Fréchet manifold (or F-manifold) we mean a separable metric space having an open cover by sets homeomorphic to s (see [14] for a summary of results on Q-manifolds and [5] for references to papers showing that all separable infinite-dimensional Fréchet spaces are homeomorphic to the Fréchet space s). We remark that in the sequel all spaces will be assumed to be separable and metric.

For any spaces A and T we use the notation

$$E(A,T) = \{f\colon A \to T \mid f \text{ is an embedding}\},$$

$$H(T) = \{f \in E(T,T) \mid f(T) = T\}.$$

(All function spaces will have the compact-open topology.)

Let X be an F-manifold, $K \subset X$ be a compact set, and let

$$p\colon H(X) \to E(K,X)$$

be defined by $p(f) = f|K$, for all $f \in H(X)$. It follows immediately from the homeomorphism extension theorem of [7] that p has a local cross-section at id_K (the identity mapping on K), i.e. there exists a neighborhood U of id_K in E(K,X) and a (not necessarily continuous) function

$$q\colon U \to H(X)$$

such that $p \circ q = id_U$. This simply means that any embedding $f\colon K \to X$ which is sufficiently close to id_K admits an extension to an element of H(X). The main result of this paper is Theorem 5.1, which implies that the local cross-section q can be canonically (i.e. continuously) constructed.

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Theorem 5.1 also implies an analogous result for Q-manifolds (with some limitations) which we describe below.

Following [2] we say that a closed subset A of a space T is a <u>Z-set</u> in T iff for each non-null and homotopically trivial open set U in T, $U\setminus A$ is also non-null and homotopically trivial. Let Y be a Q-manifold, $K\subset Y$ be a compact Z-set, and let

$$Z(K,Y) = \{f\in E(K,Y) \mid f(K) \text{ is a Z-set}\}.$$

Consider the natural function

$$p': H(Y) \to Z(K,Y)$$

defined by $p'(f) = f|K$, for all $f\in H(Y)$. The homeomorphism extension theorem of [8] implies that there exists a (not necessarily continuous) local cross-section of p' at id_K. We prove in Section 5 that we cannot always construct a continuous local cross-section of p' at id_K. (We remark that this is strictly an infinite-dimensional phenomenon associated with Q-manifolds, emphasing the intuitive fact that Q-manifolds have "dense boundaries". Indeed if M is a finite-dimensional manifold with boundary ∂M and $K = \{\text{point}\}\subset M\setminus\partial M$, then we can clearly find a neighborhood U of id_K in E(K,M) and a continuous function $q: U \to H(M)$ such that $q(f)|K = f$, for all $f\in U$.) However we can restrict ourselves to a dense subspace of Z(K,Y) for which the local cross-section can be continuously constructed.

In [6] it was shown that any Q-manifold Y satisfies $Y \cong Y\times Q$ (" $\cong$ " means "is homeomorphic to"). As in [13] this can be used to show that every Q-manifold Y contains dense F-manifolds X such that X sits in Y roughly as s sits in Q. Precisely this means that there exists a homeomorphism of Y onto $Y\times Q$ which takes X onto $Y\times s$. We call such a pair (X,Y) a <u>manifold pair</u>. It follows from [13] that if (X,Y) is a manifold pair, then any compact set $K\subset X$ is a Z-set in both X and Y. Thus we have $E(K,X)\subset Z(K,Y)$. Let

$$H^*(Y) = \{f\in H(Y) \mid f(X) = X\}$$

and let

$$p': H^*(Y) \to E(K,X)$$

be the natural function given by restriction. Then Theorem 5.1 implies that p' has a continuous local cross-section at id_K.

The remainder of the paper is concerned with applications of Theorem 5.1 to certain spaces of maps (i.e. continuous functions), embeddings, and homeomorphisms. We give below brief descriptions of some of the results obtained.

1. Applications to spaces of maps and embeddings. Let $A_0 \subset A \cap X$, where A_0 and A are compacta ($A_0 \neq A$) and X is an F-manifold. In [19] it was shown that the space C(A,X) of maps of A into X is an F-manifold. In Theorem 6.1 we prove:

(i) $E(A,X) \cong C(A,X)$, and therefore E(A,X) is also an F-manifold,

(ii) the space E_{A_0} of embeddings of A into X which are pointwise fixed on A_0 is also an F-manifold.

In Theorem 6.2 we establish some results concerning mappings and embeddings into Q-manifolds:

(i) the space of maps C(A,Y), where A is a non-degenerate compactum and Y is a Q-manifold, is an F-manifold,

(ii) using the notation above, $E(A,Y) \cong C(A,Y)$, and therefore E(A,Y) is also an F-manifold.

We remark that these techniques do not give an analogue of Theorem 6.1 (ii) for Q-manifolds.

2. Applications to homeomorphism groups of Q-manifolds. Let Y be a Q-manifold and let $K \subset Y$ be a compact Z-set. In Theorem 7.2 we use Theorem 5.1 to show that there exists a neighborhood G of K in Y, a neighborhood U of id_Y in H(Y), and a continuous deformation of U into

$$H_G(Y) = \{f \in H(Y) \mid f|G = id_G\}.$$

In Corollary 7.3 we use this to show that if Y is compact, then $H(Y \times [0,1))$ is locally contractible. In Question 7.4 we pose the open problem, is the homeomorphism group of a compact Q-manifold locally contractible?

For any Q-manifold Y it follows from the general apparatus of [20] that $H(Y) \cong H(Y)\times s$. It is shown in [22] that for the manifold pair (s,Q), $H^*(Q) \cong H^*(Q)\times s$. In Theorem 7.5 we show that for any manifold pair (X,Y), $H^*(Y) \cong H^*(Y)\times s$.

3. Applications to homeomorphisms on cells. Let $H_0(I^n)$ denote the space of all homeomorphisms of the n-cell I^n which leave the boundary ∂I^n pointwise fixed. It is well-known that $H_0(I^n)$ is contractible and Mason [23] has shown that $H_0(I^2)$ is an AR(metric). Anderson [3] has shown that $H_0(I^1) \cong s$ and it has been conjectured that $H_0(I^n) \cong s$, $n \geq 2$ [9]. In Corollary 8.2 we use Theorem 6.1 and a selection theorem of Morse [24] (concerning spaces of arcs) to obtain a short (but non-elementary) proof of Anderson's result. The lack of an analogue of Morse's selection theorem for spaces of n-cells ($n \geq 2$) is precisely why this process does not prove that $H_0(I^n) \cong s$, for all $n \geq 1$. In the connection see Question 8.3.

We make some remarks concerning notation to be used in the sequel. For products $X\times Y$ we will use p_X to denote projection onto X and for any $y\in Y$ we define (id_X,y): $X \to X\times Y$ by $(id_X,y)(x) = (x,y)$, for all $x\in X$. For products $\Pi_{i=1}^{\infty} X_i$ we use p_i to denote projection onto X_i. We will use $I(X)$ to denote the space of ambient invertible isotopies of X onto itself, i.e. $I(X)$ is the space of level-preserving homeomorphisms of $X\times I$ onto itself, where $I = [0,1]$. If $\theta\in I(X)$, then we let $\theta_t\in H(X)$ denote the homeomorphism

$$X \xrightarrow{(id_X,t)} X\times I \xrightarrow{\theta} X\times I \xrightarrow{p_X} X,$$

for all $t\in I$. If F: $X\times I \to Y$ is any function, then we let F_t: $X \to Y$ be the function

$$X \xrightarrow{(id_X,t)} X\times I \xrightarrow{F} Y,$$

for all $t\in I$.

For A a subset of a space X we let

$$H_A(X) = \{h\in H(X) \mid h|A = id_A\},$$

and if $A = \{x\}$, for some $x \in X$, we let $H_A(X) = H_x(X)$. We also use the notation $H(A,X)$ for the space of homeomorphisms of A onto X.

It should be pointed out that we will be predominately concerned with homeomorphism groups of Q-manifolds, and in these cases the homeomorphism groups are known to be topological groups [10].

2. Canonical properties of absolute retracts

The main result of this section is Theorem 2.3, where we establish a canonical property of ARs (metric). This will be used in the proof of Theorem 4.1. We first establish two easy lemmas.

Lemma 2.1. Let X *be a metric space,* Y *be an* AR, *and let* $A \subset Y$ *be compact. Then there exists a continuous function* $\alpha: X \times E(A,X) \to Y$ *such that*

$$\alpha(f(a),f) = a,$$

for all $a \in A$ *and* $f \in E(A,X)$.

Proof. Let R be the space of reals and let

$$C = \{f \in C(A,R) \mid f(a) = 0 \text{ for at most one element } a \in A\},$$

$$C_0 = \{f \in C \mid f(a) = 0 \text{ for exactly one element } a \in A\}.$$

Then C is a metric space and it is easily verified that C_0 is a closed subset of C. Define $\phi: C_0 \to A$ by $\phi(f) = a$, where $f(a) = 0$. It is easily seen that ϕ is continuous. Since Y is an AR (metric) we can extend ϕ to a continuous function $\Phi: C \to Y$. Choose any metric d for X and define a function

$$\theta: X \times E(A,X) \to C(A,R)$$

by $\theta(x,f)(a) = d(x,f(a))$. Clearly θ is continuous.

Note that the image of θ lies in C. Thus we can define $\alpha: X \times E(A,X) \to Y$ by $\alpha = \Phi \circ \theta$. For any $a \in A$ and $f \in E(A,X)$ we have

$$\alpha(f(a),f) = \Phi(\theta(f(a),f)) = \phi(\theta(f(a),f)) = a,$$

which is what we wanted.

In Lemma 2.1 we allowed the subset of X on which we constructed the extension to vary. In Lemma 2.2 we fix the subset.

Lemma 2.2. _Let_ Y _be an_ AR _and let_ $A \subset Y$ _be compact. Then there exists a continuous function_ $\beta\colon C(A,Y) \to C(Y,Y)$ _such that_ $\phi(f)|A = f$, _for all_ $f \in C(A,Y)$.

Proof. Without loss of generality Y can be assumed to be a closed subset of a convex subset of a normed linear space (see [12], page 79). The usual proof of Dugundji's generalization of Tietze's Extension Theorem yields a continuous function $\phi\colon C(A,Y) \to C(Y,C)$ such that $\phi(f)|A = f$, for all $f \in C(A,Y)$ (see [12], page 78). Let $r\colon C \to Y$ be a retraction. Then β can be defined by $\beta(f) = r \circ \phi(f)$, for all $f \in C(A,Y)$. It clearly fulfills our requirements.

Theorem 2.3. _Let_ X _be a metric space,_ Y _be an_ AR, _and let_ A _be a compact metric space. Then there exists a continuous function_

$$\phi\colon E(A,X) \times E(A,Y) \to C(X,Y)$$

such that $\phi(f,g)|f(A) = g \circ f^{-1}$, _for all_ $(f,g) \in E(A,X) \times E(A,Y)$.

Proof. Choose any $g_1 \in E(A,Y)$ and let $A_1 = g_1(A)$. It follows from Lemma 2.1 that there exists a continuous function $\alpha\colon E(A_1,X) \to C(X,Y)$ such that $\alpha(f)|f(A_1) = f^{-1}$, for all $f \in E(A_1,X)$. Using Lemma 2.2 there exists a continuous function $\beta\colon C(A_1,Y) \to C(Y,Y)$ such that $\beta(f)|A_1 = f$, for all $f \in C(A_1,Y)$. Define $\phi\colon E(A,X) \times E(A,Y) \to C(X,Y)$ by

$$\phi(f,g) = \beta(g \circ g_1^{-1}) \circ \alpha(f \circ g_1^{-1}),$$

for all $(f,g) \in E(A,X) \times E(A,Y)$. It is easily seen that ϕ fulfills our requirements.

3. _A canonical mapping replacement theorem_

Let A be completely metrizable and separable, $A_0 \subset A$ be closed, and let X be an F-manifold. In [7] it was shown that any continuous function $f\colon A \to X$, for which $f|A_0$ is a _Z-embedding_ (i.e. $f|A_0$ is a homeomorphism of A_0 onto a Z-set in X), can be approximated by Z-embeddings $g: A \to X$ for which $g|A_0 = f|A_0$. We will need the following canonical version of this for compacta.

Theorem 3.1. Let A be a compact metric space, $A_0 \subset A$ be closed, X be an F-manifold, and let

$$C^*(A,X) = \{f \in C(A,X) \mid f|A_0 \text{ is } 1\text{-}1\}.$$

Then there exists a continuous function $\phi: C^*(A,X) \times I \to C^*(A,X)$ such that for each $f \in C^*(A,X)$ and $t \in (0,1]$, $\phi_t(f) \in E(A,X)$, $\phi_0(f) = f$, and $\phi_t(f)|A_0 = f|A_0$.

Proof. It follows from [6] that there exists a homeomorphism $h: X \to X \times s$. For each $f \in C^*(A,X)$ and each integer $i > 0$ let

$$\alpha_i(f) = \mathrm{glb}\{t \mid t \in p_i \circ p_s \circ h \circ f(A)\},$$

$$\beta_i(f) = \mathrm{lub}\{t \mid t \in p_i \circ p_s \circ h \circ f(A)\}.$$

It is easy to see that we can continuously assign to each $f \in C^*(A,X)$ a unique piecewise-linear homeomorphism $\theta_i(f): I_i^0 \to I_i^0$ which satisfies $\theta_i(f)((\alpha_i(f)-1)/2) = -\frac{1}{2}$ and $\theta_i(f)(\beta_i(f)) = \frac{1}{2}$. Let $\theta(f) \in H(s)$ be defined coordinatewise by $\theta(f) = \theta_1(f) \times \theta_2(f) \times \ldots$ and note that

$$(\mathrm{id}_X \times \theta(f)) \circ h \circ f(A) \subset X \times \Pi_{i=1}^{\infty}[-\tfrac{1}{2},\tfrac{1}{2}],$$

for all $f \in C^*(A,X)$.

It follows from [1] that there exists a homeomorphism $k: s \to s_1 \times s_2$ which satisfies $k(\Pi_{i=1}^{\infty}[-\frac{1}{2},\frac{1}{2}]) \subset s_1 \times \{0\}$, where s_1 and s_2 are copies of s. Using the mapping replacement theorem of [7], as described in the comments above, we can easily construct a continuous function $F: A \times I \to s_2$ such that

(1) $F_t(A_0) = \{0\}$, for all $t \in I$,
(2) $F_0(A) = \{0\}$, and
(3) for each $t \in (0,1]$, $F_t|A \setminus A_0$ gives a closed embedding of $A \setminus A_0$ into $s_2 \setminus \{0\}$.

For each $f \in C^*(A,X)$ and $t \in I$ we let

$$\phi_t(f) = [(id_X, k\circ\theta(f))\circ h]^{-1} \circ (p_{X\times s_1} \circ (id_X, k\circ\theta(f))\circ h\circ f, F_t).$$

It can be routinely verified that ϕ fulfills our requirements.

4. A canonical homeomorphism extension theorem for Q

In [2] it was shown that if $K_1, K_2 \subset s$ are compacta and $h\colon K_1 \to K_2$ is a homeomorphism, then h can be extended to a homeomorphism of Q onto itself. We will need the following canonical version of this result. It will be used only in the proof of Theorem 5.1.

Theorem 4.1. *If A is a compact metric space, then there exists a continuous function*

$$\phi\colon E(A,s) \times E(A,s) \to H(Q)$$

such that for each $(f,g) \in E(A,s)\times E(A,s)$, $\phi(f,g)(s) = s$ *and* $\phi(f,g)|f(A) = g\circ f^{-1}$.

Proof. Using the techniques in the proof of Theorem 3.1 we can easily construct a continuous function $\alpha\colon E(A,s) \to H(Q, Q_1\times Q_2)$ such that for each $f \in E(A,s)$, $\alpha(f)(s) = s_1\times s_2$ and $\alpha(f)(f(A)) \subset Q_1\times\{0\}$. Similarly let $\beta\colon E(A,s) \to H(Q,Q_1\times Q_2)$ be a continuous function such that for each $g \in E(A,s)$, $\beta(g)(s) = s_1\times s_2$ and $\beta(g)(g(A)) \subset \{0\}\times Q_2$. (Here Q_1 and Q_2 are copies of Q and s_1, s_2 are the corresponding copies of s.)

Using Theorem 2.1 let

$$\gamma\colon E(A,Q_1) \times E(A,s_2) \to C(Q_1,s_2)$$

be a continuous function such that $\gamma(f,g)|f(A) = g\circ f^{-1}$, for all $(f,g) \in E(A,Q_1)\times E(A,s_2)$. Similarly let

$$\delta\colon E(A,Q_2) \times E(A,s_1) \to C(Q_2,s_1)$$

be a continuous function such that $\delta(f,g)|f(A) = g\circ f^{-1}$, for all $(f,g) \in E(A,Q_2)\times E(A,s_1)$.

For any $(f,g)\in E(A,s)\times E(A,s)$ it clearly follows that

$$\gamma(p_{Q_1}\circ\alpha(f),p_{Q_2}\circ\beta(g)) \in C(Q_1,s_2) \qquad \text{and}$$

$$\delta(p_{Q_2}\circ\beta(g),p_{Q_1}\circ\alpha(f)) \in C(Q_2,s_1).$$

Define $\tau(f,g)$ to be the uniquely defined homeomorphism of $Q_1\times Q_2$ onto itself such that for each $x\in Q_1$, $\tau(f,g) \mid \{x\}\times s_2 \colon \{x\}\times s_2 \to \{x\}\times s_2$ is linear and

$$\tau(f,g)(x,0) = (x,\gamma(p_{Q_1}\circ\alpha(f),p_{Q_2}\circ\beta(g))(x)).$$

Similarly let $\mu(f,g)$ be the uniquely defined homeomorphism of $Q_1\times Q_2$ onto itself such that for each $x\in Q_2$, $\mu(f,g)(s_1\times\{x\}) = s_1\times\{x\}$ and

$$\mu(f,g)(0,x) = (\delta(p_{Q_2}\circ\beta(g),p_{Q_1}\circ\alpha(f))(x),x).$$

Now for each $(f,g)\in E(A,s)\times E(A,s)$ define

$$\phi(f,g) = (\beta(g))^{-1}\circ(\mu(f,g))^{-1}\circ\tau(f,g)\circ\alpha(f),$$

which clearly fulfills our requirements.

5. A canonical homeomorphism extension theorem for manifold pairs

We now state and prove the main theorem of this paper. It is concerned with manifold pairs (X,Y) and the existence of canonical extensions of homeomorphisms between compacta in X to homeomorphisms of Y. In part 1 of the theorem we show that if a canonical homotopy is given joining the two embeddings, then a canonical homeomorphism extension exists. Part 2 of the theorem is a relative version of part 1, where the given embeddings are chosen close enough so that a canonical homotopy joining them exists, thus reducing it to part 1. As pointed out in the Introduction, parts 1 and 2 are canonical versions for compacta of the homeomorphism extension theorem of [7].

In Theorem 5.2 we prove that in some cases, the use of manifold pairs in Theorem 5.1 is necessary, even the relative version of part 2.

Theorem 5.1. <u>Let</u> (X,Y) <u>be a manifold pair</u>, A <u>be a compact metric space</u>, <u>and let</u> $C^*(A\times I,X) = \{F\in C(A\times I,X) \mid F_0, F_1 \in E(A,X)\}$.

(1) <u>There is a continuous function</u> $\phi\colon C^*(A\times I,X) \to I(Y)$ <u>such that for each</u> $F\in C^*(A\times I,X)$ <u>and</u> $t\in I$, $\phi(F)_t(X) = X$, $\phi(F)_0 = id_Y$, <u>and</u> $\phi(F)_1|F_0(A) = F_1\circ F_0^{-1}$.

(2) <u>If</u> $A\subset X$, <u>then there is a neighborhood</u> U <u>of</u> id_A <u>in</u> E(A,X) <u>and a continuous function</u> $\theta\colon U \to I(Y)$ <u>such that for each</u> $f\in E(A,X)$ <u>and</u> $t\in I$, $\theta(f)_t(X) = X$, $\theta(f)_0 = id_Y$, <u>and</u> $\theta(f)_1|A = f$.

Proof. Since (X,Y) is a manifold pair there exists a homeomorphism of Y onto Y×Q which takes X onto Y×s. We can use the techniques of the proof of Theorem 3.1 to obtain a continuous function

$$\alpha\colon C^*(A\times I,X) \to H(Y, Y\times Q_1\times Q_2)$$

such that for each $F\in C^*(A\times I,X)$, $\alpha(F)(F(A\times I))\subset Y\times Q_1\times\{0\}$ and $\alpha(F)(X) = Y\times s_1\times s_2$, where Q_1, Q_2 are copies of Q and s_1, s_2 are the corresponding copies of s. Clearly there exists an element $\phi_1'\in I(Y\times Q_1\times Q_2)$ such that $(\phi_1')_0 = id_{Y\times Q_1\times Q_2}$, $(\phi_1')_1(Y\times Q_1\times\{0\}) = Y\times Q_1\times\{(\frac12,\frac12,\ldots)\}$, and $(\phi_1')_t(x,y,z) = (x,y,z')$, for all $(x,y,z)\in Y\times Q_1\times Q_2$ and $t\in I$. For each $F\in C^*(A\times I,X)$ and $t\in I$ let

$$\phi_1(F)_t = (\alpha(F))^{-1}\circ(\phi_1')_t\circ\alpha(F),$$

which defines a continuous function $\phi_1\colon C^*(A\times I,X) \to I(Y)$ which satisfies $\phi_1(F)_t(X) = X$, for all F and t (with the corresponding choice having been made for ϕ_1'). If we then define $\beta\colon C^*(A\times I,X) \to C^*(A\times I,X)$ by

$$\beta(F)_t = \begin{cases} F_{2t}, & 0 \le t \le \frac12 \\ \phi_1(F)_{2t-1}\circ F_1, & \frac12 \le t \le 1, \end{cases}$$

then β is a continuous function such that $\beta(F)_0 = F_0$ and $\beta(F)_1\cap\beta(F)_0 = \emptyset$, for all $F\in C^*(A\times I,X)$.

Using Theorem 3.1 we can construct a continuous function

$$\gamma\colon C^*(A\times I,X) \to E(A\times[-1,2],X)$$

such that for each $F\in C^*(A\times I,X)$, $\gamma(F)_0 = \beta(F)_0$ and $\gamma(F)_1 = \beta(F)_1$. Let $h\colon Y\to Y\times Q$ be a homeomorphism such that $h(X) = Y\times s$ and use the open embedding theorem of [15] to obtain an open embedding $k\colon Y\times(Q\setminus\{p\})\to Q$, where $p\in Q\setminus s$. The apparatus of [13] enables us to make adjustments on k to additionally require that $k(Y\times s) = k(Y\times(Q\setminus\{p\}))\cap s$. Thus for each $F\in C^*(A\times I,X)$ we have $k\circ h\circ\gamma(F)\in E(A\times[-1,2],s)$.

It follows from [4] that there exists a homeomorphism of Q onto $Q\times[-1,2]$ which takes s onto $s\times[-1,2]$. Without loss of generality we can assume that $A\subset s$. Then apply Theorem 4.1 to obtain a continuous function

$$\delta\colon E(A\times[-1,2],s) \to H(Q,Q\times[-1,2])$$

such that for each $F\in E(A\times[-1,2],s)$, $\delta(F)(s) = s\times[-1,2]$ and $\delta(F)\circ F(a,t) = (a,t)$, for all $(a,t)\in A\times[-1,2]$.

Let $G = k(Y\times(Q\setminus\{p\}))$ and note that for each $F\in C^*(A\times I,X)$, $\delta(k\circ h\circ\gamma(F))(G)$ is an open subset of $Q\times[-1,2]$ containing $A\times[-1,2]$. Let d be any metric for Q and for each $F\in C^*(A\times I,X)$ we let

$$\epsilon(F) = \mathrm{lub}\{t \mid N_t(A)\times[-1,2] \subset \delta(k\circ h\circ\gamma(F))(G)\},$$

where $N_t(A) = \{x\in Q \mid d(A,x)<t\}$. It is clear that $\epsilon(F)$ depends continuously on F.

Using motions only in the $[-1,2]$-direction we can easily construct a continuous function

$$\tau\colon C^*(A\times I,X) \to I(Q\times[-1,2])$$

such that for each $F\in C^*(A\times I,X)$ and $t\in I$, $\tau(F)_t(s\times[-1,2]) = s\times[-1,2]$, $\tau(F)_0 = \mathrm{id}_{Q\times[-1,2]}$, $\tau(F)_t|(Q\times[-1,2])\setminus(N_{\epsilon(F)/2}(A)\times[-1,2]) = \mathrm{id}$, and $\tau(F)_1(x,0) = (x,1)$, for all $x\in N_{\epsilon(F)/4}(A)$.

For each $F\in C^*(A\times I,X)$ let $\mu(F) = \delta(k\circ h\circ\gamma(F))\circ k\circ h$ and define $\phi_2\colon C^*(A\times I,X) \to I(Y)$ by

$$\phi_2(F)_t = \begin{cases} ((\mu(F))^{-1}\circ\tau(F)_t\circ\mu(F), & \text{on } (\mu(F))^{-1}(N_{\epsilon(F)/2}(A)\times[-1,2]) \\ \text{id}, & \text{on } Y\setminus(\mu(F))^{-1}(N_{\epsilon(F)/2}(A)\times[-1,2]), \end{cases}$$

for all $F\in C^*(A\times I,X)$ and $t\in I$. For each $F\in C^*(A\times I,X)$ and $t\in I$ define

$$\phi(F)_t = \begin{cases} \phi_2(F)_{2t}, & 0 \leq t \leq \frac{1}{2} \\ (\phi_1(F)_{2t-1})^{-1}, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

It is easily checked that ϕ fulfills the requirements for (1) of the theorem.

For (2) we first recall that X can be regarded as an open subset of s [21]. Then since A is compact we can choose U so that for any $f\in U$, the straight-line homotopy joining id_A to f lies in X. This gives an element of $C^*(A\times I,X)$ which depends continuously on f and we can therefore apply (1).

Theorem 5.2. <u>Choose</u> $K = \{\text{point}\}\subset Q$ <u>and let</u>

$$p\colon H(Q) \to Z(K,Q)$$

<u>be the natural function given by restriction. Then there does not exist a continuous local cross-section of</u> p <u>at</u> id_K.

Proof. We will assume the contrary, that is there exists a neighborhood U of id_K in $Z(K,Q)$ and a continuous function $q\colon U \to H(Q)$ such that $p\circ q = \text{id}_U$, and we seek a contradiction. This implies that there exists a point $x_0\in Q$, a neighborhood G of x_0 in Q, and a continuous function $\phi\colon G \to H(Q)$ such that $\phi(x)(x_0) = x$, for every element $x\in G$. Using techniques from [15] there exist Hilbert cubes Q_1,Q_2 such that $Q = Q_1\cup Q_2$, $Q_1\cap Q_2$ is a Hilbert cube which is collared in each of Q_1 and Q_2, $\text{Bd}(Q_1)\cap\text{Bd}(Q_2) = Q_1\cap Q_2$, $x_0\in\text{Int}(Q_1)\subset Q_1\subset G$.

Let $\alpha = \phi|Q_1$ and use the fact that Q is homogeneous to get a continuous function $\beta\colon Q_2 \to H(Q)$ such that $\beta(x)(x_0) = x$, for all $x\in Q_2$. Now note that [24] implies that $H_{x_0}(Q)$ is contractible. Let $\gamma = \beta^{-1}\circ\alpha|Q_1\cap Q_2$, where by β^{-1} we mean $\beta^{-1}(x) = (\beta(x))^{-1} \in H(Q)$, for all $x \in Q$. The homeomorphism extension theorem of [2] assures us that Q_2 can be viewed as a cone over $Q_1 \cap Q_2$.

Since $\gamma(Q_1 \cap Q_2) \subset H_{x_0}(Q)$, the contractibility of $H_{x_0}(Q)$ implies that γ can be extended to a continuous function δ: $Q_2 \to H_{x_0}(Q)$. Then we can define θ: $Q \to H(Q)$ by setting

$$\theta(x) = \begin{cases} \alpha(x), & \text{for } x \in Q_1 \\ \beta \circ \delta(x), & \text{for } x \in Q_2, \end{cases}$$

which is a continuous function satisfying $\theta(x)(x_0) = x$, for all $x \in Q$.

Choose $x_1 \neq x_0$ and note that g: $Q \to Q$, defined by $g(x) = \phi(x)(x_1)$, gives a fixed-point free map of Q into itself. This gives our contradiction.

6. Mappings into manifolds

In Theorem 6.1 we prove that spaces of embeddings of compacta into F-manifolds are themselves F-manifolds. To treat those embeddings which are fixed on proper closed subsets we will have to apply Theorem 5.1. In Theorem 6.2 we prove that spaces of maps and embeddings of compacta into Q-manifolds are F-manifolds.

Theorem 6.1. *Let A be a compact metric space, X be an F-manifold, and let $A_0 \subset A \cap X$ be a proper closed subset of A (that is $A_0 \neq A$). Then*

(1) $E(A,X) \cong C(A,X)$ (*and therefore by* [19] $E(A,X)$ *is an F-manifold*) *and*

(2) $E_{A_0}(A,X)$ *is an F-manifold.*

Proof. It follows from [19] that $C(A,X)$ is an F-manifold. Let d be a metric for A and for each integer $n > 0$ let

$$A_n = \{f \in C(A,X) \mid d(x,y) \geq \tfrac{1}{n} \text{ and } f(x) = f(y), \text{ for some } x,y \in A\}.$$

It is easily seen that each A_n is closed in $C(A,X)$. It also easily follows from the definition of a Z-set and Theorem 3.1 that each A_n is a Z-set in $C(A,X)$, for all $n > 0$. The main result of [4] implies that coutable unions of Z-sets in F-manifolds are negligible. That is,

$$C(A,X) \cong C(A,X)\setminus \bigcup_{n=1}^{\infty} A_n.$$

Since $E(A,X) = C(A,X)\setminus \bigcup_{n=1}^{\infty} A_n$ we have established (1).

We now show that $E_{A_0}(A,X)$ is an F-manifold. As pointed out in the Introduction we can choose a Q-manifold Y such that (X,Y) is a manifold pair. Using Theorem 5.1 there exists an open set U in $E(A_0,X)$ containing id_{A_0} and a continuous function

$$\phi\colon U \to H^*(Y)$$

such that for each $f\in U$, $\phi(f)|K = f$. Let

$$p\colon E(A,X) \to E(A_0,X)$$

be given by restriction, that is $p(f) = f|A_0$ for all $f\in E(A,X)$. Then p is continuous and $p^{-1}(U)$ is a neighborhood of id_A. We can easily define a homeomorphism

$$\theta\colon p^{-1}(U) \to U \times E_{A_0}(A,X)$$

by setting $\theta(f) = (p(f),\phi(p(f))^{-1}\circ f)$, for all $f\in p^{-1}(U)$. Since $E(A_0,X)$ is now known to be an F-manifold we can choose $U = s$. Thus $E_{A_0}(A,X)\times s$ is homeomorphic to an open subset of the F-manifold $E(A,X)$, and it is therefore an F-manifold.

Now choose $f_0\in E_{A_0}(A,X)$, $x_0\in A\setminus A_0$, and put $A' = A_0\cup\{x_0\}$. Assume $A'\subset X$ and without loss of generality assume $A' \neq A$, as otherwise we would obviously have

$$E_{A_0}(A,X) \cong X\setminus A_0 \cong X.$$

Let $V\subset E_{A_0}(A',X)$ be a neighborhood of $f_0|A'$ and let $\psi\colon V \to H^*(Y)$ be a continuous function such that for each $f \in V$, $\psi(f)|A' = f$. Let

$$q: E_{A_0}(A,X) \to E_{A_0}(A',X)$$

be given by restriction and use an argument similar to that given above to show that $q^{-1}(V) \cong V \times E_{A'}(A,X)$. Since $E_{A_0}(A',X)$ is an F-manifold we must have $V \cong V \times s$ (see [6]), hence $q^{-1}(V) \cong q^{-1}(V) \times s$. From above it follows that $q^{-1}(V) \times s$ is an F-manifold, thus $q^{-1}(V)$ is an F-manifold. Since $q^{-1}(V)$ contains f_0 we are done.

Theorem 6.2. *Let A be a non-discrete compact metric space and let Y be a Q-manifold. Then*

(1) C(A,Y) *is an F-manifold and*

(2) E(A,Y) = C(A,Y)

Proof. We first show that $C(A,Y \times I)$ is an F-manifold and then use the fact $Y \cong Y \times I$ to conclude that C(A,Y) is an F-manifold. We use a technique from [16] that was used to prove that the space of maps from a non-discrete compactum to a compact finite-dimensional manifold with smooth interior is an F-manifold. Following [16] we say that a closed subset K of a space M has *Isotopy Property* A if there exists a homotopy $H: M \times I \to M$ such that (1) $H_0 = id_M$, (2) for each $t > 0$, H_t is a homeomorphism of M onto a closed subset of M missing K, and (3) if $u > t$, then $H_u(M) \subset Int(H_t(M))$. It is shown in [16] that if K is a closed subset of M such that (1) K has Isotopy Property A, (2) $M \times s$ is an F-manifold, and (3) $M \setminus K$ is an F-manifold, then M itself is an F-manifold.

We will apply this technique with $M = C(A, Y \times I)$ and $K = M \setminus C(A, Y \times (0,1))$. Clearly K is closed in M and it follows from [20] that $M \times s$ and $M \setminus K$ are F-manifolds. Since $Y \times I$ can be isotopically deformed into $Y \times (0,1)$ we can use this to prove that K has Isotopy Property A. Thus C(A,Y) is an F-manifold.

To see that E(A,Y) = C(A,Y) we use the idea of Theorem 6.1 and write $C(A,Y) \setminus E(A,Y) = \bigcup_{n=1}^{\infty} A_n$, where

$$A_n = \{f \in C(A,Y) \mid d(x,y) \geq 1/n \text{ and } f(x) = f(y), \text{ for some } x,y \in A\}.$$

To see that each A_n is a Z-set in C(A,Y) we can use the argument given in Theorem 6.1 by first choosing an F-manifold $X \subset Y$ such that (X,Y) is a manifold pair, and then noting that Y can be continuously deformed into X with a "small" motion, thereby enabling us to use Theorem 3.1.

7. Deforming homeomorphism groups of Q-manifolds

The main result of this section is Theorem 7.2, where we show how to locally deform the homeomorphism group of any Q-manifold to the identity on some neighborhood of a given compact Z-set. In Corollary 7.3 we use this to obtain results on the local contractibility of homeomorphism groups of certain Q-manifolds. In Theorem 7.5 we obtain a general factor result and in Corollary 7.6 we use it to generalize a result of Keesling [22]. For Theorem 7.2 we will need the following technical lemma.

Lemma 7.1. *Let X be a Q-manifold and let* $\alpha: H(X\times Q) \to H(X\times Q)$ *be defined by*

$$\alpha(f)(x,(t_i)) = (x',(t_1',t_2,t_3',t_4,\ldots)),$$

where $f(x,(t_1,t_3,\ldots)) = (x',(t_1',t_3',\ldots))$. *Then there exists a continuous function* $\phi: H(X\times Q) \times I \to H(X\times Q)$ *such that* $\phi(f)_0 = \alpha(f)$ *and* $\phi(f)_1 = f$, for all $f \in H(X\times Q)$.

Proof. Our proof is similar to an argument used by Barit [11] to prove that $H_K(Q)$ is contractible, for any Z-set $K \subset Q$. It is also typical of the apparatus developed by Wong [28] and used by Renz [25]. For each integer $n \geq 1$ define $\alpha_n: H(X\times Q) \to H(X\times Q)$ by

$$\alpha_n(f)(x,(t_i)) = (x',(t_1',t_2',\ldots,t_{2n+1}',t_{2n+2},t_{2n+3}',t_{2n+4},\ldots)),$$

where

$$f(x,(t_1,t_2,\ldots,t_{2n+1},t_{2n+3},\ldots)) = (x',(t_1',t_2',\ldots,t_{2n+1}',t_{2n+3}',\ldots)).$$

Choose $\beta \in I(Q)$ which satisfies $\beta_0 = id_Q$ and

$$\beta_1((t_i)) = (t_1,t_4,t_2,t_6,t_3,t_8,t_5,t_{10},t_7,\ldots),$$

i.e. β_1 puts the first even coordinate in the second odd coordinate position. Define ϕ^1: $H(X\times Q) \times I \to H(X\times Q)$ by

$$\phi_t^1(f) = (id_X\times\beta_t^{-1})\circ\alpha(f)\circ(id_X\times\beta_t).$$

Clearly ϕ^1 is continuous, $\phi_0^1 = \alpha$, and $\phi_1^1 = \alpha_1$.

The idea is to repeat this process to obtain a sequence $\{\phi^n\}_{n=1}^{\infty}$ of homotopies ϕ^n: $H(X\times Q) \times I \to H(X\times Q)$ defined by

$$\phi_t^n(f) = (id_X\times id_{I_1}\times\ldots\times id_{I_{2n}}\times{}_n\beta_t^{-1})\circ\alpha_{n-1}(f)\circ(id_X\times id_{I_1}\times\ldots\times id_{I_{2n}}\times{}_n\beta_t),$$

where ${}_n\beta_t$ is the natural analogue of β_t applied to $I_{2n+1} \times I_{2n+2} \times \ldots$. Clearly each ϕ^n is continuous, $\phi_0^n = \alpha_{n-1}$, and $\phi_1^n = \alpha_n$.

Define ϕ: $H(X\times Q) \times I \to H(X\times Q)$ so that ϕ_1 = id and for each integer $n > 0$, $\phi|H(X\times Q) \times [\frac{n-1}{n}, \frac{n}{n+1}]$ is just ϕ^n linearly scaled down from $H(X\times Q) \times I$ to $H(X\times Q) \times [\frac{n-1}{n}, \frac{n}{n+1}]$. It can be verified that ϕ is continuous.

Theorem 7.2. Let Y be a Q-manifold and let K be a compact Z-set in Y. Then there exists a neighborhood $G\subset Y$ of K and a neighborhood U of id_Y in H(Y) for which there exists a continuous function ϕ: $U\times I\to H(Y)$ which satisfies $\phi_0(f) = f$ and $\phi_1(f)|G = id_G$, for all $f\in U$.

Proof. Since K is a Z-set it follows from [13] that there exists a homeomorphism h: $Y\to Y\times Q$ such that

$$h(K) \subset Y \times I_1 \times \{0\} \times I_3 \times \{0\} \times \ldots .$$

Let

$$X' = Y \times I_1 \times I_2^0 \times I_3 \times I_4^0 \times \ldots$$

and not that by [13], $(X',Y\times Q)$ is a manifold pair. It is easily seen that if α is the function of Lemma 7.1, then $\alpha(f)(X') = X'$, for all $f\in H(Y\times Q)$.

Using the result of Lemma 7.1, applied to $Y\times Q$, and then transferring it back to Y (as h induces a natural homeomorphism of H(Y) onto $H(Y\times Q)$), we can then get a continuous function ${}_1\theta$: $H(Y) \times I \to H(Y)$ such that for all $f \in H(Y)$, ${}_1\theta_0(f) = f$ and ${}_1\theta_1(f)(K) \subset X = h^{-1}(X')$. Note also from the construction given in Lemma 7.1 we can additionally require that ${}_1\theta_t(\mathrm{id}_Y) = \mathrm{id}_Y$, for all $t \in I$.

Now apply Theorem 5.1 to find an open set V in $E(K,X)$ containing id_K and a continuous function θ_2: $V \to I(Y)$ such that for every $f \in E(K,X)$, $\theta_2(f)_0 = \mathrm{id}_Y$ and $\theta_2(f)_1|K = f$. In the construction of θ_2 no provision was made for $\theta_2(\mathrm{id}_K)_t = \mathrm{id}_Y$, for all $t \in I$. Since $I(Y)$ is a topological group we can make an obvious adjustment of θ_2 to additionally require this property.

Let us define a function β: $H(Y) \to E(K,X)$ by $\beta(f) = {}_1\theta_1(f)|K$. Then β is continuous and $\beta(\mathrm{id}_Y) = \mathrm{id}_K$. Thus $\beta^{-1}(V)$ is an open subset of $H(Y)$ containing id_Y. Set $U_1 = \beta^{-1}(V)$ and define ${}_1\phi$: $U_1 \times I \to H(Y)$ by

$$ {}_1\phi_t(f) = \begin{cases} {}_1\theta_{2t}(f), & \text{for } 0 \le t \le \frac{1}{2} \\ (\theta_2(\beta(f))_{2t-1})^{-1} \circ {}_1\theta_1(f), & \text{for } \frac{1}{2} \le t \le 1, \end{cases} $$

for all $f \in H(Y)$, and note that ${}_1\phi_0(f) = f$, ${}_1\phi_1(f)|K = \mathrm{id}_K$.

We need now to show that there exists a compact Q-manifold M and an open embedding g: $M \times [0,1) \to Y$ such that $K \subset g(M \times \{0\})$. Using the properties (1) there exists a homeomorphism of Y onto $Y \times [0,1]$ taking K into $Y \times \{0\}$ and (2) there exists an open embedding of $Y \times [0,1)$ into Q, we can find an open set $W \subset Q$ and an open embedding u: $W \to Y$ such that $K \subset u(W)$. It follows from [15] that there exists a locally compact polyhedron L' and a homeomorphism v: $W \to L' \times Q$. Since $u^{-1}(K)$ is a Z-set in W we may adjust v so that

$$ v \circ u^{-1}(K) \subset L' \times I_1 \times \{0\} \times I_3 \times \{0\} \times \dots . $$

Thus there exists a compact polyhedron $L \subset L'$ such that

$$ v \circ u^{-1}(K) \subset L \times I_1 \times \{0\} \times I_3 \times \{0\} \times \dots . $$

Now let i: $L \to Y$ be defined by

$$ i(x) = u \circ v^{-1}(x,(0,0,\dots)), $$

for all $x \in L$. This is a homeomorphism of L onto a Z-set in Y. From [15] it follows that there exists an open embedding $\tilde{i}: L\times Q\times[0,1)\to Y$ such that $\tilde{i}(x,0,0) = i(x)$, for all $x\in L$. Define

$$j: \tilde{i}(L\times(I_1\times\{0\}\times I_3\times\{0\}\times\ldots)\times\{0\}) \to u\circ v^{-1}(L\times(I_1\times\{0\}\times I_3\times\{0\}\times\ldots)) \text{ by}$$

$$j\circ\tilde{i}(x,(t_1,0,t_3,0,\ldots),0) = u\circ v^{-1}(x,(t_1,0,t_3,0,\ldots)),$$

for all $x\in L$ and $(t_1,t_3,\ldots)\in I_1\times I_3\times\ldots$. Then j is a homeomorphism between Z-sets which is homotopic to the identity (in Y). Using [8] we can extend j to a homeomorphism $\tilde{j}: Y \to Y$. Then $\tilde{j}\circ\tilde{i}: L\times Q\times[0,1)\to Y$ is an open embedding such that $K\subset j\circ i(L\times Q\times\{0\})$. Let $M = L\times Q$ and $g = \tilde{j}\circ\tilde{i}$ to fulfill our requirements.

Let $G = g(M\times[0,\frac{1}{2}))$ and use the standard Alexander trick (see [17], page 321) to obtain a continuous function

$${}_2\phi: H_{g(M\times\{0\})}(Y) \times I \to H(Y)$$

such that for all $f\in H_{g(M\times\{0\})}(Y)$ and $t\in I$, ${}_2\phi_0(f) = f$, ${}_2\phi_1(f)|G = id_G$, and ${}_2\phi_t(f)|Y\setminus g(M\times[0,1)) = id$.

Now apply the construction of ${}_1\phi$ to the compact Z-set $g(M\times\{0\})$ to obtain a continuous function

$${}_3\phi: U_3 \times I \to H(Y)$$

(where U_3 is a neighborhood of id_Y in $H(Y)$) such that for each $f\in U_3$, ${}_3\phi_0(f) = f$ and ${}_3\phi_1(f)|g(M\times\{0\}) = id$. Put $U = U_3$ and let $\phi: U\times I\to H(Y)$ be defined by

$$\phi_t(f) = \begin{cases} {}_3\phi_{2t}(f), & \text{for } 0 \le t \le \frac{1}{2} \\ {}_2\phi_{2t-1}({}_3\phi_1(f)), & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

It is easily seen that ϕ fulfills our requirements.

Corollary 7.3. <u>Let</u> Y <u>be</u> <u>a</u> <u>compact</u> <u>Q-manifold</u>. <u>Then</u> $H(Y\times[0,1))$ <u>is</u> <u>locally</u> <u>contractible</u>.

Proof. This follows immediately from Theorem 7.2 by (1) deforming a neighborhood of id_Y to $H_{Y\times\{0\}}(Y\times[0,1))$ and (2) using the Alexander trick.

In light of the recent result that H(M) is locally contractible [18], where M is a compact finite-dimensional manifold, it would be reasonable to ask if the factor [0,1) in Corollary 8.3 could be omitted.

Question 7.4. *If* Y *is a compact Q-manifold, then is* H(Y) *locally contractible (and in fact an F-manifold)?*

Theorem 7.5. *If* (X,Y) *is any manifold pair, then*

$$H^*(Y) \cong H^*(Y) \times s.$$

Proof. Choose any $x_0 \in X$ and let

$$p: H^*(Y) \to X$$

be defined by $p(f) = f(x_0)$, for all $f \in H^*(Y)$. It follows from [15] that there exists an embedding $g: Q \to Y$ such that $Bd(g(Q)) = g(W)$ and $g(0) = x_0$, where

$$W = \{(x_i) \in Q \mid x_1 = 1\}.$$

Using the apparatus of [13] we can additionally require that $g(s) = g(Q\setminus W) \cap X$. It follows from [2] that there exists a homeomorphism of Q onto itself which takes $s \cup W$ onto s. Thus Theorem 4.1 can be applied to obtain a continuous function

$$\phi: s \to H_W(Q)$$

such that $\phi(x)(0) = x$ and $\phi(x)(s) = s$, for all $x \in s$. Put $U = g(s)$ and we use ϕ to obtain a continuous function

$$\theta: U \to H^*(Y)$$

so that $\theta(x)(x_0) = x$, for all $x \in U$. Define

$$\psi\colon p^{-1}(U) \to U \times H^*_{x_0}(U)$$

by

$$\psi(f) = (p(f),(\phi(p(f)))^{-1}\circ f),$$

where $H^*_{x_0}(U) = H_{x_0}(U) \cap H^*(U)$. It is clear that ϕ is a homeomorphism. Since the choice of x_0 was arbitrary it follows that $H^*(Y)$ is a separable manifold modeled on $H^*_{x_0}(Y)\times s$. (We assume, without loss of generality, that Y is connected. Thus $H^*_{x_0}(Y) \cong H^*_{x_1}(Y)$, for all $x_0,x_1 \in X$.)

It can be shown that if M is a separable metric space, then any separable manifold N modeled on M×s admits an s-factor. This does not appear explicitly in [26], but it can be established from the apparatus given there. In the notation of [26], one would have to prove that N satisfies Property P_s locally and then apply Theorem 3.5 of [26]. Using this result it then follows that $H^*(Y)$ admits an s-factor.

Corollary 7.6. (Keesling [22]). For the manifold pair (s,Q), $H^*(Q) = H^*(Q)\times s$.

8. Homeomorphisms on cells

In Corollary 8.2 we apply Theorem 6.1 to obtain a short proof of Anderson's theorem that $H_0(I^1) \cong s$ [3]. In Question 8.3 we pose an open question which, if answered affirmatively, would imply that $H_0(I^n) \cong s$, for all $n > 0$. The proof of Corollary 8.2 involves a selection theorem of Morse which we summarize in Theorem 8.1. For a detailed treatment see [20].

Let A be a compact metric space, $A_0 \subset A$ be a closed subset, and let (X,d) be a metric space for which $A_0 \subset X$. By $S_{A_0}(A,X)$ we mean the collection of all subsets B of X for which there exists a homeomorphism $h\colon A \to B$ satisfying $h|A_0 = id$. Topologize $S_{A_0}(A,X)$ by defining, for each pair $B,C \in S_{A_0}(A,X)$,

$$p(B,C) = \mathrm{glb}\{d(h,id_B) \mid h\colon B \to C \text{ is a homeomorphism, } h|A_0 = id\},$$

where $d(h, id_B) = \text{lub}\{d(h(x),x) \mid x \in B\}$. Then $(S_{A_0}(A,X),\rho)$ is easily seen to be a metric space. Define

$$p: E_{A_0}(A,X) \to S_{A_0}(A,X)$$

by $p(f) = f(A)$, for all $f \in E_{A_0}(A,X)$. Then p is continuous and each fiber $p^{-1}(B)$ is homeomorphic to $H_{A_0}(A)$. The following is essentially due to Morse [24].

Theorem 8.1. (Morse [24]). _If_ $A = I^1$ _and_ $A_0 = \partial I^1$, _then there exists a continuous cross-section of_ p.

Corollary 8.2. (Anderson [3]). $H_0(I^1) \cong s$.

Proof. Assume $I = I^1 \subset s$ and let

$$p: E_{\partial I}(I,s) \to S_{\partial I}(I,s)$$

be defined as above. Theorem 8.1 gives a continuous function

$$q: S_{\partial I}(I,s) \to E_{\partial I}(I,s)$$

such that $p \circ q = id$. Then

$$\phi: E_{\partial I}(I,s) \to S_{\partial I}(I,s) \times H_0(I),$$

defined by $\phi(f) = (p(f),(q(p(f)))^{-1} \circ f)$, gives a homeomorphism. Theorem 6.1 implies that $E_{\partial I}(I,s)$ is an F-manifold and it is easily seen that $E_{\partial I}(I,s)$ is contractible (in fact, Theorem 5.1 implies that $E(I,s) \cong s \times E_{\partial I}(I,s) \cong E_{\partial I}(I,s)$). It then follows from [21] that $E_{\partial I}(I,s) \cong s$. By a trick of West [27] we have $H_0(I) \times s \cong s$.

To see that $H_0(I) \cong H_0(I) \times s$ (and to avoid the use of [20]) we can easily obtain a homeomorphism ψ of $H_0(I)$ onto

$$(0,1) \times H_0([0,\tfrac{1}{2}]) \times H_0([\tfrac{1}{2},1])$$

by defining $\psi(f) = (f(\tfrac{1}{2}),f_1,f_2)$, where f_1 is just $f|[0,\tfrac{1}{2}]$ linearly transposed to an element of $H_0([0,\frac{1}{2}])$ and f_2 is similarly defined. Repeating this process infinitely often we can easily obtain a homeomrophism of $H_0(I)$ onto

$$[(0,1)\times(0,1/2)\times(0,1/3)\times\ldots]\times[H_0([1/2,1])\times H_0([1/3,1/2])\times H_0([1/4,1/3])\times\ldots].$$

We remark that if Theorem 8.1 were true for I^n, $n > 1$, then the proof of Corollary 8.2 would apply to prove that $H_0(I^n) \cong s$.

Question 8.3. *Assume* $I^n \subset s$ *and let*

$$p: E_{\partial I^n}(I^n,s) \to S_{\partial I^n}(I^n,s)$$

be defined by $p(f) = f(I^n)$. *Does* p *have a continuous cross-section* $(n > 1)$?

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